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Explicit Order 1.5 Runge-Kutta Scheme for Solutions of Itô Stochastic Differential Equations

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Abstract

In this paper, a new approximation scheme is developed for the solutions of Itô stochastic differential equations by employing a Runge-Kutta (RK) method. Both drift and dispersion functions are iteratively evaluated by 4 kinds of terms to achieve higher order accuracy of approximation than conventional methods. The proposed RK scheme is organized by an explicit 4-stage structure and is of order 1.5. The asymptotic efficiency of the proposed scheme is also established. The simulation results are shown for supporting the validity of the approximation scheme.

1. Introduction

The problem considered in this paper is to develop a numerical method for solutions of the following stochastic differential equations (SDEs) of Itô type :

$$dx(t) = f(x)dt + g(x)dw(t), \quad x(0) = \bar{x}_0 \in R^d. \tag{1}$$

Here $w(t)$ is a one-dimensional standard Wiener process, i.e., (i) $P(w(0)=0) = 1$, (ii) $\mathrm{E}\{w(t)\} = 0$, $^\forall t \in [0, \infty[$ and (iii) $\mathrm{E}\{w(t)w(s)\} = \min(t, s)$ where $\mathrm{E}\{\cdot\}$ denotes the mean value of $(\cdot)$. The functions f and g mapping from R^d to R^d are assumed to be smooth enough so that SDE (1) has a unique strong solution, i.e., they satisfy both Lipschitz and linear growth conditions. The stochastic integral implied on the r.h.s. of Eq.(1) is an Itô integral. The approximation is evaluated at points of regular partitions of the interval $[0, T]$, namely at points $(0, h, 2h, \ldots, Nh)$ where N is a natural number and $h = T/N$.

For developing an approximation scheme for SDEs, it is useful to introduce the method of discretization, which is analogous to the numerical integration for deterministic differential equations (DDEs), for example, Taylor series schemes and Runge-Kutta (RK) type schemes [1], [2]. This will enable us to obtain a stronger kind of convergence and precise information about the error. Thus various kinds of numerical schemes for SDEs have been proposed based on Taylor series and RK methods. These include [3]-[7] and the references therein. From practical viewpoints, Taylor schemes have a disadvantage that they need analytical expressions of the derivatives of the coefficient functions and in general, it is not easy to evaluate the higher order derivatives of nonlinear functions. On the other hand, Runge-Kutta schemes require only the evaluations of the coefficient functions. The RK schemes for SDEs differ from those for DDEs in that they involve the iterative evaluations

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of not only drift function f but also dispersion function g. Rümelin [6] has shown that it is necessary to choose the coefficients of RK method for SDEs carefully in order that the resulting approximations converge to the solution of the right equations, and that, for one-dimensional equations, the traditional RK methods for SDEs of Itô type converge only semilinearly in h. Newton [8] developed an asymptotically efficient order 1.0 RK scheme which has explicit form and is organized by 2-stage evaluations of f and 4-stage evaluations of g. However the order conditions for the scheme in [8] are not clear. Saito and Mitsui [9] presented a way to improve the accuracy for 3-stage RK scheme. The explicit scheme in [9] is a stochastic version of Heun's method and guarantees order 1.5 accuracy. However to use the order 1.5 RK scheme in [9], the 1st order derivatives of f and g and the 2nd order derivative of g must be given analytically. This is the same kind of disadvantage as Taylor schemes possess.

In this paper, an efficient Runge-Kutta (RK) scheme of order 1.5 accuracy is presented for Itô SDE (1). The present RK scheme is organized by 4 kinds of terms to avoid the iterative evaluations of the derivatives of the coefficients functions f and g. This paper is organized as follows: Section 2 gives order 1.5 approximation of the solution to SDE (1) and show the structure of RK scheme. Based on the structure, an explicit 4-stage RK scheme is derived and the analytical results on approximation error are shown in Section 3. Section 4 shows the results of simulation studies for six examples. Finally, Section 5 provides conclusions.

2. Runge-Kutta Approximation for SDEs

In order to discuss numerical approximation methods for stochastic differential equations, we introduce a notation to formulate Itô-Taylor expansion. For a positive integer m, let $\alpha = (j_1, \cdots, j_m)$ where $j_i = 0, 1$, $(i = 1, \cdots, m)$, then multiple Itô stochastic integrals are expressed as follows [3, 9]:

$$I_\alpha(y; t_0, t) = \int_{t_0}^{t} \int_{t_0}^{\tau_m} \cdots \int_{t_0}^{\tau_2} y(x(\tau_1)) dW^{(j_1)}(\tau_1) \cdots dW^{(j_{m-1})}(\tau_{m-1}) dW^{(j_m)}(\tau_m)$$

where the definitions of $dW^{(j)}$s are given by

$$dW^{(j)}(t) = \begin{cases} dt & (j = 0), \\ dw(t) & (j = 1), \end{cases}$$

In particular, more simple expression $I_\alpha(t_0, t)$ is used for the case $y \equiv 1$.

Then the order 1.5 Itô-Taylor expansion of the solution to Eq.(1) is expressed as follows:

$$\begin{aligned} x(t_{n+1}) = & x(t_n) + g(x(t_n)) I_{(1)}(t_n, t_{n+1}) + f(x(t_n)) I_{(0)}(t_n, t_{n+1}) \\ & + [Gg](x(t_n)) I_{(1,1)}(t_n, t_{n+1}) + [Fg](x(t_n)) I_{(0,1)}(t_n, t_{n+1}) \\ & + [Gf](x(t_n)) I_{(1,0)}(t_n, t_{n+1}) + [G^2 g](x(t_n)) I_{(1,1,1)}(t_n, t_{n+1}) + R(t_n, h) \end{aligned} \qquad (2)$$

where the differential operators F and G are defined by

$$F(\cdot) = \sum_{j=1}^{d} f_j \frac{\partial}{\partial x_j}(\cdot) + \frac{1}{2}\sum_{j,k=1}^{d} g_j g_k \frac{\partial^2}{\partial x_j \partial x_k}(\cdot) = (\cdot)_x(f) + \frac{1}{2}(\cdot)_{xx}(g,g),$$

$$G(\cdot) = \sum_{j=1}^{d} g_j \frac{\partial}{\partial x_j}(\cdot) = (\cdot)_x(g).$$

and the "remainder" $\rho(t_n, h)$ satisfies that $\mathrm{E}\{ (R(t_n, h))^2 \} = O(h^4)$. In Eq.(2), we set $t_n = nh$ and then the integrals of the type $I_{(j_1,\cdots,j_m)}(t_n, t_{n+1})$ are calculated as follows:

$$\begin{aligned} I_{(1)}(t_n,t_{n+1}) &= \Delta W_n, \quad I_{(0)}(t_n,t_{n+1}) = h \\ I_{(1,1)}(t_n,t_{n+1}) &= \frac{1}{2}(\,\Delta W_n^2 - h\,) \\ I_{(0,1)}(t_n,t_{n+1}) &= \Delta W_n h \; - \; I_{(1,0)}(t_n,t_{n+1}) \\ I_{(1,1,1)}(t_n,t_{n+1}) &= \frac{1}{6}(\,\Delta W_n^3 - 3\Delta W_n h\,) \end{aligned}$$

where ΔW_n denotes the increment of one-dimensional Wiener processes and is simulated by the sample values of normal random numbers $\xi_n \in N(0,1)$, i.e., $\Delta W_n = \xi_n \sqrt{h}$. According to Platen and Wagner [3, Chapter 10.4], the integrals $I_{(0,1)}(t_n, t_{n+1})$ and $I_{(1,0)}(t_n, t_{n+1})$ are set as,

$$\begin{aligned} I_{(0,1)}(t_n,t_{n+1}) &= \frac{h}{2}\left(\Delta W_n - \frac{1}{\sqrt{3}}\Delta \tilde{W}_n\right) \\ I_{(1,0)}(t_n,t_{n+1}) &= \frac{h}{2}\left(\Delta W_n + \frac{1}{\sqrt{3}}\Delta \tilde{W}_n\right) \end{aligned}$$

where $\Delta\tilde{W}_n = \tilde{\xi} h^{1/2}$, $\tilde{\xi}_n \in N(0,1)$ and the random numbers ξ_n and $\tilde{\xi}_n$ are independent each other.

Thus the order 1.5 Itô-Taylor expansion is expressed as follows:

$$\begin{aligned} x(t_{n+1}) =& x(t_n) + g(x(t_n))\Delta W_n + f(x(t_n))h + [Gg](x(t_n))\frac{\Delta W_n^2 - h}{2} \\ &+ \Big([Fg](x(t_n)) + [Gf](x(t_n))\Big)\frac{\Delta W_n}{2}h \\ &+ \Big([Gf](x(t_n)) - [Fg](x(t_n))\Big)\Delta\tilde{W}_n \frac{h}{2\sqrt{3}} \\ &+ [G^2 g](x(t_n))\frac{\Delta W_n^3 - 3\Delta W_n h}{6} + R(t_n,h), \end{aligned} \tag{3}$$

As shown in Eq.(3), the order 1.5 expansion contains the "bias" terms and the terms in the increments of two kinds of Wiener processes. In this paper, the maximum order of convergence is achieved by introducing the following s-stage Runge-Kutta approximation structure:

$$\bar{x}_{n+1} = \bar{x}_n + h\sum_{i=1}^{s} b_i k_i + \Delta W_n \sum_{i=1}^{s} \bar{b}_i \bar{k}_i + \frac{\Delta\tilde{W}_n}{\sqrt{3}}\sum_{i=1}^{s}\tilde{b}_i\tilde{k}_i + \sqrt{\nu h}\sum_{i=1}^{s}\hat{b}_i\hat{k}_i. \tag{4a}$$

Here $\bar{x}_n$ is the numerical solution at the point $t = nh$, and the parameters k_i, $\bar{k}_i$, $\tilde{k}_i$ and $\hat{k}_i$ are the function values such that

$$\begin{cases} k_i = f\Big(\bar{x}_n + h\sum_{j=1}^{i-1} a_{ij}k_j + \Delta W_n \sum_{j=1}^{i-1} \bar{a}_{ij}\bar{k}_j + \frac{\Delta \tilde{W}_n}{\sqrt{3}} \sum_{j=1}^{i-1} \tilde{a}_{ij}\tilde{k}_j\Big) \\ \bar{k}_i = g\Big(\bar{x}_n + h\sum_{j=1}^{i-1} a_{ij}k_j + \Delta W_n \sum_{j=1}^{i-1} \bar{a}_{ij}\bar{k}_j + \sqrt{\nu h} \sum_{j=1}^{i-1} \hat{a}_{ij}\hat{k}_j\Big) \\ \tilde{k}_i = g\Big(\bar{x}_n + h\sum_{j=1}^{i-1} a_{ij}k_j + \sqrt{\nu h} \sum_{j=1}^{i-1} \hat{a}_{ij}\hat{k}_j\Big) \\ \hat{k}_i = g\Big(\bar{x}_n + \sqrt{\nu h} \sum_{j=1}^{i-1} \hat{a}_{ij}\hat{k}_j\Big) \end{cases} \tag{4b}$$

where ν is a positive constant. The parameters $\{b_i\}$, $\{\bar{b}_i\}$, $\{\tilde{b}_i\}$ and $\{\hat{b}_i\}$ for $i = 1, \cdots, s$, in (4a), and $\{a_{ij}\}$, $\{\bar{a}_{ij}\}$, $\{\tilde{a}_{ij}\}$ and $\{\hat{a}_{ij}\}$ for $i = 1, \cdots, s$ $(j = 1, \cdots, i-1)$ in (4b), are determined so that the numerical solution obtained by the scheme (7) has the assigned accuracy.

3. Order 1.5 Runge-Kutta Scheme for Itô SDEs

In the previous section, a new framework of the numerical scheme described by (4) is proposed for Itô stochastic equations to achieve higher order accuracy. The goal of this section is to develop an explicit order 1.5 Runge-Kutta scheme by using the approximation structure (4). Then the conditions on the parameters in (4) are summarized by the following lemma.

Lemma 1. (order conditions) *In order to achieve order 1.5 accuracy by approximation structure (4), the constants $\{b_i\}$, $\{\bar{b}_i\}$, $\{\tilde{b}_i\}$, $\{\hat{b}_i\}$ $(i = 1, \ldots, s)$ and $\{a_{ij}\}$, $\{\bar{a}_{ij}\}$, $\{\tilde{a}_{ij}\}$, $\{\hat{a}_{ij}\}$ $(i = 1, \ldots, s,\ j = 1, \ldots, i-1)$ in (4) are constrained by the following 22 conditions:*

$$\begin{array}{llll}
(c.1) & \sum_i \bar{b}_i = 1, & (c.2) & \sum_i \bar{b}_i\bar{c}_i = \frac{1}{2}, \\
(c.3) & \sum_i \hat{b}_i\hat{c}_i = -\frac{1}{2\nu}, & (c.4) & \sum_i b_i = 1, \\
(c.5) & \sum_i \bar{b}_i\bar{c}_i^2 = \frac{1}{3}, & (c.6) & \sum_i \bar{b}_i\hat{c}_i^2 = -\frac{1}{2\nu}, \\
(c.7) & \sum_{ij} \bar{b}_i\bar{a}_{ij}\bar{c}_j = \frac{1}{6}, & (c.8) & \sum_{ij} \bar{b}_i\hat{a}_{ij}\hat{c}_j = -\frac{1}{2\nu}, \\
(c.9) & \sum_i \bar{b}_i c_i = \frac{1}{2}, & (c.10) & \sum_i b_i\bar{c}_i = \frac{1}{2}, \\
(c.11) & \sum_i b_i\tilde{c}_i = \frac{1}{2}, & (c.12) & \sum_i \tilde{b}_i c_i = -\frac{1}{2} \\
(c.13) & \sum_i \tilde{b}_i\hat{c}_i^2 = -\frac{1}{2\nu}, & (c.14) & \sum_{ij} \bar{b}_i\bar{a}_{ij}\hat{c}_j = 0, \\
(c.15) & \sum_i \bar{b}_i\hat{c}_i = 0, & (c.16) & \sum_i \bar{b}_i\bar{c}_i\hat{c}_i = 0, \\
(c.17) & \sum_i \tilde{b}_i = 0, & (c.18) & \sum_i \tilde{b}_i\hat{c}_i = 0, \\
(c.19) & \sum_{ij} \tilde{b}_i\hat{a}_{ij}\hat{c}_j = 0, & (c.20) & \sum_i \hat{b}_i = 0, \\
(c.21) & \sum_{ij} \hat{b}_i\hat{a}_{ij}\hat{c}_j = 0, & (c.22) & \sum_i \hat{b}_i\hat{c}_i^2 = 0.
\end{array}$$

where $c_i = \sum_{j=1}^{i-1} a_{ij}$, $\bar{c}_i = \sum_{j=1}^{i-1} \bar{a}_{ij}$, $\tilde{c}_i = \sum_{j=1}^{i-1} \tilde{a}_{ij}$ *and* $\hat{c}_i = \sum_{j=1}^{i-1} \hat{a}_{ij}$. *In* (*c*.1) $\sim$ (*c*.22), $\sum_i$ *and* $\sum_{ij}$ *denote* $\sum_i = \sum_{i=1}^{s}$ *and* $\sum_{ij} = \sum_{i=1}^{s} \sum_{j=1}^{i-1}$.

The conditions in *Lemma 1* are easily derived by comparing the coefficients of terms in the Itô-Taylor expansion of order 1.5 (Eq.(2)) and the Taylor expansion of the numerical scheme (4) (see *Appendix*).

Thus, the parameters in (4) must be determined by solving the order conditions in *Lemma 1* algebraically and will be described below.

3.1. Explicit 4-Stage Runge-Kutta Scheme of Order 1.5

Since the order conditions are simultaneous nonlinear algebraic equations, the solution in general is not unique. Here we propose the following 4-stage scheme as an example that satisfies the order conditions in *Lemma 1*. This scheme is first summarized below and will be derived in detail in Section 3.2.

The numerical solution of (1) is obtained by

$$\begin{aligned} \bar{x}_{n+1} = \bar{x}_n + & \left(\frac{1}{6}k_1 - \frac{2}{9}k_2 + \frac{8}{9}k_3 + \frac{1}{6}k_4\right) h + \left(\frac{1}{6}\bar{k}_1 - \frac{2}{9}\bar{k}_2 + \frac{8}{9}\bar{k}_3 + \frac{1}{6}\bar{k}_4\right) \Delta W_n \\ & + \frac{1}{\sqrt{3}} \left(\frac{1}{6}\tilde{k}_1 - \frac{2}{9}\tilde{k}_2 + \frac{8}{9}\tilde{k}_3 - \frac{5}{6}\tilde{k}_4\right) \Delta \tilde{W}_n + \left(-\frac{1}{18}\hat{k}_2 + \frac{8}{9}\hat{k}_3 - \frac{5}{6}\hat{k}_4\right) \sqrt{3h}, \end{aligned} \tag{5a}$$

where k_i, $\bar{k}_i$, $\tilde{k}_i$ and $\hat{k}_i$ ($i = 1 \sim 4$) are obtained from (4b) by using the following parameters:

$$\begin{aligned} \{a_{ij}\} = \{\bar{a}_{ij}\} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 \\ 1/3 & -2 & 8/3 & 0 \end{pmatrix}, \\ \{\tilde{a}_{ij}\} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \{\hat{a}_{ij}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -13/32 & 5/32 & 0 & 0 \\ -7/24 & 1/8 & 1/6 & 0 \end{pmatrix}. \end{aligned} \tag{5b}$$

3.2. Derivation Procedure of Scheme (5)

We now find parameters satisfying the order conditions in *Lemma 1*. Taking into account the conditions (c.1), (c.2), (c.4), (c.9), (c.10) and (c.11), we set $b_i = \bar{b}_i$, $a_{ij} = \bar{a}_{ij}$ and $c_i = \tilde{c}_i$. Then the number of the conditions on the parameters decreases to 18. If we set the stage number of the RK scheme as $s = 4$, the number of parameters which must be chosen, $b_i = \bar{b}_i$, $\tilde{b}_i$, $\hat{b}_i$, $c_i = \bar{c}_i = \tilde{c}_i$, $\hat{c}_i$, $a_{ij} = \bar{a}_{ij}$, $\tilde{a}_{ij}$ and $\hat{a}_{ij}$, becomes 27. Then there are 9 degree of freedom in the choice of the parameters which is possible to choose arbitrarily (certain choices of these parameters have to be excluded if they cause any later steps to be impossible because of zero divisors). Notice that if $s = 3$, the number of parameters to be chosen becomes 16 and it is impossible to choose 16 parameters so that 18 conditions on parameters are satisfied.

Thus we set as $s = 4$. Comparing (c.19) and (c.21), we also set $\tilde{b}_3 = \hat{b}_3$ and $\tilde{b}_4 = \hat{b}_4$. Then the number of the conditions on the parameters decreases to 17, and the number of parameters which must be chosen becomes 25 (there are 8 degree of freedom in the choice of the parameters). With the above procedures, the following steps are carried out:

Step 1 Solve (c.2), (c.5) and (c.16) with respect to b_2c_2, b_3c_3 and b_4c_4. (the solutions b_2c_2, b_3c_3 and b_4c_4 are described by the parameters c_2, c_3, c_4, $\hat{c}_2$, $\hat{c}_3$ and $\hat{c}_4$)
Step 2 Solve (c.6), (c.15) and (c.16) with respect to $b_2\hat{c}_2$, $b_3\hat{c}_3$ and $b_4\hat{c}_4$. (the solutions $b_2\hat{c}_2$, $b_3\hat{c}_3$ and $b_4\hat{c}_4$ are described by the parameters c_2, c_3, c_4, $\hat{c}_2$, $\hat{c}_3$ and $\hat{c}_4$)
Step 3 Solve (c.12), (c.13) and (c.18) with respect to $\tilde{b}_2$, $\tilde{b}_3$ and $\tilde{b}_4$. (the solutions $\tilde{b}_2$, $\tilde{b}_3$ and $\tilde{b}_4$ are described by the parameters c_2, c_3, c_4, $\hat{c}_2$, $\hat{c}_3$ and $\hat{c}_4$)
Step 4 Solve (c.3) with respect to $\hat{b}_2$. (let $\hat{b}_2'$ be the solution described by the parameters c_2, c_3, c_4, $\hat{c}_2$, $\hat{c}_3$ and $\hat{c}_4$)
Step 5 Solve (c.22) with respect to $\hat{b}_2$. (let $\hat{b}_2''$ be the solution described by the parameters c_2, c_3, c_4, $\hat{c}_2$, $\hat{c}_3$ and $\hat{c}_4$)
Step 6 Solve $\hat{b}_2' - \hat{b}_2'' = 0$ with respect to $\hat{c}_2$. By setting the free parameter ν as $\nu = 3$, we have $\hat{c}_2 = -1$
Step 7 Solve $(b_3c_3) \times \hat{c}_3 - (b_3\hat{c}_3) \times c_3 = 0$ with respect to c_3 and substitute the solution c_3 into b_2c_2, b_3c_3, b_4c_4, $b_2\hat{c}_2$ and $b_4\hat{c}_4$.
Step 8 Solve $(b_4c_4) \times \hat{c}_4 - (b_4\hat{c}_4) \times c_4 = 0$ with respect to $\hat{c}_3$ and substitute the solution $\hat{c}_3$ into b_2c_2, b_3c_3, b_4c_4, $b_2\hat{c}_2$ and $b_4\hat{c}_4$.
Step 9 Obtain b_2, b_3 and b_4 by $b_2 := (b_2c_2)/c_2$, $b_3 := (b_3c_3)/c3$ and $b_4 := (b_4c_4)/c_4$.
Step 10 Substitute $\hat{c}_2 = -1$ into b_2, b_3, b_4, $\tilde{b}_2$, $\hat{b}_2$, $\tilde{b}_3(=\hat{b}_3)$ and $\tilde{b}_4(=\hat{b}_4)$.
Step 11 Solve (c.1), (c.17) and (c.20) with respect to b_1, $\tilde{b}_1$ and $\hat{b}_1$.

As the results of the above steps, we obtain 13 parameters by solving 13 equations in (c.1), (c.2), (c.3), (c.5), (c.6), (c.12), (c.13), (c.14), (c.15), (c.16), (c.17), (c.20) and (c.22), i.e., $\hat{c}_2$ is determined as $\hat{c}_2 = -1$ and the 12 parameters b_1, b_2, b_3, b_4, $\tilde{b}_1$, $\hat{b}_1$, $\tilde{b}_2$, $\hat{b}_2$, $\tilde{b}_3(=\hat{b}_3)$ $\tilde{b}_4(=\hat{b}_4)$ $c_3(=\tilde{c}_3)$ and $\hat{c}_3$ are described by the 3 parameters c_2, c_4 and $\hat{c}_4$. Since there are 8 degree of freedom in the choice of parameter, we make the following choice:

$$c_2 = \tfrac{1}{2}, \qquad c_4 = 1, \qquad \hat{c}_4 = 0.$$

by using 3 degree of freedom. Substituting these parameters into the solutions for 13 equations, we have

$$\begin{aligned}
&b_1 = \bar{b}_1 = 1/6, \quad \tilde{b}_1 = 1/6, \quad \hat{b}_1 = 0, b_2 = \bar{b}_2 = -2/9, \quad \tilde{b}_2 = -2/9, \quad \hat{b}_2 = -1/18,\\
&b_3 = \bar{b}_3 = 8/9, \quad \tilde{b}_3 = \hat{b}_3 = 8/9, b_4 = \bar{b}_4 = 1/6, \quad \tilde{b}_4 = \hat{b}_4 = -5/6,\\
&a_{21} = \bar{a}_{21} = \tilde{a}_{21} = 1/2, \quad \hat{a}_{21} = -1,\\
&c_3 = \bar{c}_3 = \tilde{c}_3 = 1/2, \quad \hat{c}_3 = -1/4, c_4 = \bar{c}_4 = \tilde{c}_4 = 1, \quad \hat{c}_4 = 0.
\end{aligned}$$

Now we must choose 9 parameters $a_{32} = \bar{a}_{32}$, $a_{42} = \bar{a}_{42}$, $a_{43} = \bar{a}_{43}$, $\tilde{a}_{32}$, $\tilde{a}_{42}$, $\tilde{a}_{43}$, $\hat{a}_{32}$, $\hat{a}_{42}$ and $\hat{a}_{43}$ by using the 4 conditions (c.7), (c.8), (c.14) and (c.19), i.e., there are 5 degrees of freedom in the choice of 9 parameters.
Step 12 Solve (c.7) and (c.14) with respect to $\alpha_1 = b_4a_{42} + b_3a_{32}$ and $\beta_1 = b_4a_{43}$.
Then we have $a_{43} = 8/3$.
Step 13 Solve (c.8) and (c.19) with respect to $\alpha_2 = \hat{a}_{32}\hat{c}_2$ and $\beta_2 = \hat{a}_{43}\hat{c}_3 + \hat{a}_{42}\hat{c}_2$.
Then we have $\hat{a}_{32} = 5/32$. By using 2 degrees of freedom, we set as $a_{32} = 1/4$ and $\hat{a}_{43} = 1/6$. Then we have $a_{31} = 1/4$, $\hat{a}_{42} = 1/8$. Finally, by using the remaining 3 degrees of freedom we set $\tilde{a}_{31} = \tilde{a}_{41} = \tilde{a}_{42} = 0$. Then we have $\tilde{a}_{32} = 1/2$ and $\tilde{a}_{43} = 1$.

3.3. Error Evaluations

This section is devoted to show the statistical properties of errors resulting from the proposed RK scheme.

Theorem 1. *Assume that the following conditions hold:* (A.1) f *is Lipschitz continuous with all derivatives of* f *up to order 3 inclusive polynomial growth.* (A.2) g *is Lipschitz continuous with all derivatives of* g *up to order 4 inclusive polynomial growth. Then the error evaluation for the proposed numerical scheme (5) is given by*

$$\mathrm{E}\left\{ \| x_n - \bar{x}_n \|^2 \,|\, x_{n-1} = \bar{x}_{n-1} = X_{n-1} \right\} = O(h^4), \tag{6}$$

and

$$\sup_{1 \le n \le N} \mathrm{E}\left\{ \| x_n - \bar{x}_n \|^2 \right\} = O(h^3), \tag{7}$$

where $x_n = x(t_n)$ *is the true value and* $\bar{x}_n$ *is the numerical solution at the point* $t = t_n$.

proof From the assumptions (A.1), (A.2) and the order conditions (c.1) to (c.22), the terms in (5) are evaluated as follows:
(i) The term of the parameters k_i $(1 \le i \le 4)$:

$$\begin{aligned} & h\left(\frac{1}{6}k_1 - \frac{2}{9}k_2 + \frac{8}{9}k_3 + \frac{1}{6}k_4\right) \\ & = h\Big(f(\bar{x}_n) + \frac{1}{2}\Delta W_n [f_x(g)](\bar{x}_n) - \frac{1}{2}\frac{\Delta \tilde{W}_n}{\sqrt{3}}[f_x(g)](\bar{x}_n)\Big) + \rho, \end{aligned} \tag{8a}$$

where the remainder ρ satisfies

$$\mathrm{E}\left\{\|h^{-2}\rho\|^p\right\} \le M, \quad (M : \text{ bounded constant}). \tag{8b}$$

(ii) The term of the parameters $\bar{k}_i$ $(1 \le i \le 4)$:

$$\begin{aligned} & \Delta W_n\left(\frac{1}{6}\bar{k}_1 - \frac{2}{9}\bar{k}_2 + \frac{8}{9}\bar{k}_3 + \frac{1}{6}\bar{k}_4\right) \\ & = \Delta W_n\Big(g(\bar{x}_n) + \frac{h}{2}[g_x(f)](\bar{x}_n) + \frac{\Delta W_n}{2}[g_x(g)](\bar{x}_n) \\ & \quad + \frac{\Delta W_n^2}{6}[g_x(g_x(g))](\bar{x}_n) - \frac{h}{4}[g_{xx}(g,g)](\bar{x}_n) \\ & \quad + \frac{\Delta W_n^2}{6}[g_{xx}(g,g)](\bar{x}_n) - \frac{h}{2}[g_x(g_x(g))](\bar{x}_n)\Big) + \bar{\rho}, \end{aligned} \tag{9a}$$

where the remainder $\bar{\rho}$ satisfies

$$\mathrm{E}\left\{\|h^{-2}\bar{\rho}\|^p\right\} \le \bar{M}, \quad (\bar{M} : \text{ bounded constant}). \tag{9b}$$

(iii) The term of the parameter $\tilde{k}_i$ $(1 \le i \le 4)$:

$$\begin{aligned} & \frac{\Delta W_n}{\sqrt{3}}\left(\frac{1}{6}\tilde{k}_1 - \frac{2}{9}\tilde{k}_2 + \frac{8}{9}\tilde{k}_3 - \frac{5}{6}\tilde{k}_4\right) \\ & = -\frac{\Delta W_n}{\sqrt{3}}\Big(\frac{h}{2}[g_x(f)](\bar{x}_n) + \frac{h}{4}[g_{xx}(g,g)](\bar{x}_n)\Big) + \tilde{\rho}, \end{aligned} \tag{10a}$$

where the remainder $\tilde{\rho}$ satisfies

$$\mathrm{E}\left\{\|h^{-2}\tilde{\rho}\|^p\right\} \le \tilde{M}, \quad (\tilde{M} : \text{ bounded constant}). \tag{10b}$$

(iv) The term of the parameters $\hat{k}_i$ $(1 \le i \le 4)$:

$$\sqrt{3h}\left(-\frac{1}{18}\hat{k}_2+\frac{8}{9}\hat{k}_3-\frac{5}{6}\hat{k}_4\right) = -\frac{h}{2}[g_x(g)](\bar{x}_n)+\hat{\rho}, \tag{11a}$$

where the remainder $\hat{\rho}$ satisfies

$$\mathrm{E}\left\{\|h^{-2}\hat{\rho}\|^p\right\} \le \hat{M}, \quad (\hat{M}\ :\ \text{bounded constant}). \tag{11b}$$

Substituting the evaluations (8), (9), (10) and (11) into (5), it follows that

$$\begin{aligned}\bar{x}_{n+1} = \bar{x}_n &+ [g](\bar{x}_n)\Delta W_n + [f](\bar{x}_n)h + [g_x(g)](\bar{x}_n)\frac{\Delta W_n^2}{2} - [g_x(g)](\bar{x}_n)\frac{h}{2}\\ &+ [f_x(g)](\bar{x}_n)\frac{\Delta W_n}{2}h + [g_x(f)](\bar{x}_n)\frac{\Delta W_n}{2}h\\ &+ \left([f_x(g)](\bar{x}_n) - [g_x(f)](\bar{x}_n) - \frac{1}{2}[g_{xx}(g,g)](\bar{x}_n)\right)\frac{h}{2\sqrt{3}}\Delta\tilde{W}_n\\ &+ [g_{xx}(g,g)](\bar{x}_n)\frac{\Delta W_n^3}{6} - [g_{xx}(g,g)](\bar{x}_n)\frac{\Delta W_n h}{4}\\ &+ [g_x(g_x(g))](\bar{x}_n)\frac{\Delta W_n^3}{6} - [g_x(g_x(g))](\bar{x}_n)\frac{\Delta W_n h}{2}\\ &+ \rho + \bar{\rho} + \tilde{\rho} + \hat{\rho}.\end{aligned} \tag{12}$$

In order to evaluate the local error, we set $x_n = \bar{x}_n$. By subtracting (12) from (2), the local error $x_{n+1} - \bar{x}_n$ is given by

$$x_{n+1} - \bar{x}_{n+1} = R(t_n,h) + \rho(t_n,h),$$

where $\rho(t_n,h) = -\rho - \bar{\rho} - \tilde{\rho} - \hat{\rho} \sim O(h^2)$. Thus the local error is evaluated by (6).

On the other hand, the recursive equation of approximation error is obtained by subtracting (12) from (2):

$$\begin{aligned}x_{n+1} - \bar{x}_{n+1} = x_n - \bar{x}_n &+ \Big(g(x_n) - g(\bar{x}_n)\Big)\Delta W_n + \Big(f(x_n) - f(\bar{x}_n)\Big)h\\ &+ \Big([Gg](x_n) - [Gg](\bar{x}_n)\Big)\frac{\Delta W_n^2 - h}{2}\\ &+ \Big([Fg](x_n) - [Fg](\bar{x}_n)\Big)\left(\Delta W_n - \frac{\Delta\tilde{W}_n}{\sqrt{3}}\right)\frac{h}{2}\\ &+ \Big([Gf](x_n) - [Gf](\bar{x}_n)\Big)\left(\Delta W_n + \frac{\Delta\tilde{W}_n}{\sqrt{3}}\right)\frac{h}{2}\\ &+ \Big([G^2g](x_n) - [G^2g](\bar{x}_n)\Big)\frac{\Delta W_n^3 - 3\Delta W_n h}{6}\\ &+ R(t_n,h) + \rho(t_n,h).\end{aligned} \tag{13}$$

Along the similar way to Mil'shtein (1985), it is easy to show that

$$\mathrm{E}\{\,\|\,\rho(t_n,h)\,\|^2\,\} = O(h^4).$$

Squaring both sides of (13), taking expectation and using the conditions (A.1) and (A.2), we have

$$e_{n+1} \leq (1 + M_1 h + M_2 h^2 + M_3 h^3)\, e_n + M_4 h^4, \tag{14}$$

where $e_n = \mathrm{E}\{\|x_n - \bar{x}_n\|^2\}$ and M_1, M_2, M_3 and M_4 are bounded positive constants.

Now we set as

$$1 + M_1 h + M_2 h^2 + M_3 h^3 = 1 + hL$$

Then taking into account $x_0 = \bar{x}_0$, it follows that

$$\begin{aligned} e_1 &\leq M_4 h^4, \\ e_2 &\leq (1 + hL)e_1 + M_4 h^4, \\ e_3 &\leq (1 + hL)e_2 + M_4 h^4 \\ &\leq \{(1 + hL)^2 + (1 + hL) + 1\} M_4 h^4. \end{aligned}$$

Thus we have

$$e_n \leq \frac{(1 + hL)^n - 1}{hL} M_4 h^4 \leq \frac{\exp(nhL) - 1}{L} M_4 h^3, \quad \text{for } n = 0, 1, 2, \cdots, N. \tag{15}$$

The evaluation (15) shows that the global error is evaluated by (7).

Corollary 1. *Under the conditions in Theorem 1,* for any $q < 3/2$,

$$\mathrm{P}\left(\| h^{-q} (x_T - \bar{x}_N) \|^2 \rightarrow 0 \right) = 1. \tag{16}$$

This Corollary is obtained as a direct consequence of Theorem 1.

Theorem 2. *If the conditions in Theorem 1 hold, then $\{\bar{x}_n\}$ given by (5) is order 1.5 asymptotically efficient approximation in the sense that for any $\mathcal{P}_{N_i}$-adapted sequence $\{\bar{\bar{x}}_n\}$,*

$$\liminf_{i \to \infty} \frac{\mathrm{E}\left\{ \| h^{-3/2}(x_T - \bar{\bar{x}}_{N_i}) \|^2 \mid \mathcal{P}_{N_i} \right\} + 1}{\mathrm{E}\left\{ \| h^{-3/2}(x_T - \bar{x}_{N_i}) \|^2 \mid \mathcal{P}_{N_i} \right\} + 1} \geq 1, \qquad \text{w.p.1}, \tag{17}$$

where $\mathcal{P}_N = \sigma(w_h, w_{2h} \cdots, w_{Nh}, \tilde{w}_h, \tilde{w}_{2h}, \cdots, \tilde{w}_{Nh})$ and $\{N_i\}$ is a sequence of natural numbers with the property that N_{i+1}/N_i is a natural number greater than 1.

proof Let $\eta_n = h^{-3/2} | x_n - \bar{x}_n |$, then from the approximation error equation (16),

$$\eta_{n+1} \leq (1 + C_1)\eta_n + C_2 h^{1/2} + C_3 h^{-1/2} \Delta \tilde{W}_n, \quad \eta_0 = 0 \tag{18}$$

is derived, where C_i $(i = 1, 2, 3)$ are bounded positive constants.

Now we consider an equation

$$\zeta_{n+1} = (1 + C_1)\zeta_n + C_2 h^{1/2} + C_3 h^{-1/2} \Delta \tilde{W}_n, \qquad \zeta_0 = \eta_0. \tag{19}$$

Comparing (18) and (19), it is easy to see that $\eta_n \leq \zeta_n$. Moreover let $\hat{\zeta}_n$ be the approximation for ζ_n as follows:

$$\hat{\zeta}_{n+1} = (1 + C_1)\hat{\zeta}_n + C_3 h^{-1/2} \Delta \tilde{W}_n, \qquad \hat{\zeta}_0 = \eta_0. \tag{20}$$

Then the error between ζ_n and $\hat{\zeta}_n$ is given by

$$\zeta_{n+1} - \hat{\zeta}_{n+1} = (1 + C_1)(\zeta_n - \hat{\zeta}_n) + C_2 h^{1/2}. \tag{21}$$

Then it follows that

$$\sup_{1 \le n \le N} \mathrm{E}_w\{\|h^{-1/2}(\zeta_n - \hat{\zeta}_n)\|\} < \infty, \tag{22}$$

where E_w is expectation with respect to Wiener measure on $(C(0,T), \mathcal{F}_T)$. This shows that

$$\mathrm{E}_w\{\|\zeta_n - \hat{\zeta}_n\| \,|\, \mathcal{P}_N\} \to 0, \qquad \text{w.p.1.} \tag{23}$$

According to the evaluation (23), ζ_n converges to $\hat{\zeta}_n$ with zero mean. This also shows that η_n converges to nominal distribution with zero mean.

In order to prove Theorem 2, it suffices to show that

$$\mathrm{E}\left\{ h^{-3/2}\,(\, x_T \, - \, \bar{x}_{N_i}\,) \,\,|\mathcal{P}_{N_i} \right\} \to 0, \qquad \text{w.p.1.} \tag{24}$$

But, this is the direct consequence of the convergence of ζ_n.

4. Simulation Results

The numerical scheme developed here (JK95) are tested along with Euler-Maruyama scheme (EM) [10], FRKI scheme [8], ERKI scheme [8] and improved 3-stage RK scheme (SM92) [9] on the following six examples:

Example 1.

$$dx(t) = \; 0.5x(t)dt \; + \; 0.5x(t)dw(t) \;\; \text{with} \;\; x(0) = 0.5. \tag{25}$$

The exact solution of Eq.(25) is given by

$$x(t) \; = \; 0.5 \exp\Big(0.375t \; + \; 0.5w(t)\Big).$$

Example 2.

$$dx(t) = \; -x(t)dt \; + \; 0.5x(t)dw(t) \;\; \text{with} \;\; x(0) = 0.5 \tag{26}$$

The exact solution of Eq.(26) is given by

$$x(t) \; = \; \exp\Big(-1.125t \; + \; 0.5w(t)\Big).$$

Example 3.

$$dx(t) = \; a^2 \sin x \cos^3 x dt \; + \; a \cos^2 x dw(t) \;\; \text{with} \;\; x(0) = x_0. \tag{27}$$

The exact solution of Eq.(27) is given by

$$x(t) = \arctan(\, \tan x_0 + aw(t)\,)$$

Here the constants are set as $x_0 = 1$ and $a = 0.5$.

Example 4.

$$dx(t) = \frac{1}{2}a^2 m x^{(2m-1)} dt \, + \, a x^m dw(t) \ \text{ with } \ x(0) = x_0 \ \text{ and } \ m \neq 1. \tag{28}$$

The exact solution of Eq.(28) is given by

$$x(t) = \left(x_0^{(1-m)} - a(m-1)w(t) \right)^{1/(m-1)}$$

Here the constants are set as $x_0 = 0.55$, $a = 0.175$ and $m = 3$.

Example 5.

$$dx(t) = -\frac{1}{2}a^2 x dt \, + \, a\sqrt{1-x^2} dw(t) \ \text{ with } \ x(0) = x_0. \tag{29}$$

The exact solution of Eq.(29) is given by

$$x(t) = \sin(\, \arcsin x_0 + a w(t)\,)$$

Here the constants are set as $x_0 = 0.5$ and $a = 0.1$.

Example 6.

$$dx(t) = -(\, a + b^2 x\,)(1 - x^2) dt \, + \, b(1-x^2) dw(t) \ \text{ with } \ x(0) = x_0. \tag{31}$$

The exact solution of Eq.(30) is given by

$$x(t) = \frac{1 - x_0 + (1 + x_0)\exp(-2at + 2bw(t))}{-1 + x_0 + (1 + x_0)\exp(-2at + 2bw(t))}$$

Here the constants are set as $x_0 = 0.5$, $a = 0.01$ and $b = 0.5$.

In each case the mean square error at the final time ($T = 1$) is estimated in the following way.

$$e = \frac{1}{10000} \sum_{k=1}^{10000} \left(x_N^k - \bar{x}_N^k \right)^2 \tag{32}$$

where superscript k is the k-th trajectory of each solution.

The results of the simulations for for *Example 1* to *5* are shown in Tables 1–5 or Figs.1–5 (with corresponding numbers).

Table 1. mean square error (32) for *Example 1*

step size	Maruyama	FRKI	ERKI	SM92	JK95
2^{-4}	2.79730D−01	1.33581D−02	6.59533D−04	1.91705D−04	8.53110D−05
2^{-5}	2.58327D−01	3.76326D−03	1.51947D−04	3.85904D−05	1.90340D−05
2^{-6}	2.42182D−01	1.21037D−03	3.48999D−05	8.49142D−06	4.00875D−06
2^{-7}	2.38207D−01	3.29701D−04	8.74286D−06	2.02393D−06	1.04901D−06
2^{-8}	2.19879D−01	7.75466D−05	1.98311D−06	4.45569D−07	2.59120D−07

Table 2. mean square error (32) for *Example 2*

step size	Maruyama	FRKI	ERKI	SM92	JK95
2^{-4}	4.26720D−02	7.42069D−04	1.62883D−04	2.90636D−05	6.32108D−05
2^{-5}	3.95724D−02	3.75553D−04	3.81603D−05	5.73658D−06	7.88483D−06
2^{-6}	3.56530D−02	3.49930D−05	8.43144D−06	1.25503D−06	8.49554D−07
2^{-7}	3.30714D−02	1.41883D−05	2.26138D−06	3.12478D−07	1.50958D−07
2^{-8}	3.03934D−02	4.87565D−06	5.95255D−07	8.14165D−08	2.56973D−08

Table 3. mean square error (32) for *Example 3*

step size	Maruyama	FRKI	ERKI	SM92	JK95
2^{-4}	1.47861D−02	1.19161D−04	1.83534D−05	4.38800D−06	2.97771D−06
2^{-5}	1.40227D−02	3.34161D−05	3.11974D−06	9.13093D−07	5.86901D−07
2^{-6}	1.38596D−02	1.02832D−05	6.42669D−07	2.19538D−07	1.20125D−07
2^{-7}	1.33299D−02	3.18002D−06	1.29582D−07	5.09370D−08	2.53636D−08
2^{-8}	1.31645D−02	1.05660D−06	2.98921D−08	1.23156D−08	5.94687D−09

Table 4. mean square error (32) for *Example 4*

step size	Maruyama	FRKI	ERKI	SM92	JK95
2^{-4}	7.53823D−05	4.73792D−07	4.90029D−08	8.87116D−09	1.27805D−09
2^{-5}	6.73521D−05	1.43911D−07	1.09741D−08	1.72453D−09	2.58817D−10
2^{-6}	5.78771D−05	2.42528D−08	8.14532D−10	1.29275D−10	5.77899D−11
2^{-7}	5.73209D−05	8.27717D−09	2.61677D−10	4.26315D−11	1.41254D−11
2^{-8}	5.55686D−05	3.21024D−09	6.66556D−11	1.01614D−11	3.86335D−12

Table 5. mean square error (32) for *Example 5*

step size	Maruyama	FRKI	ERKI	SM92	JK95
2^{-4}	———	3.41768D−08	9.82386D−09	1.76210D−11	7.19488D−10
2^{-5}	1.10586D−04	1.57038D−08	2.04421D−09	5.51404D−12	1.32852D−10
2^{-6}	1.03636D−04	7.56102D−09	3.89050D−10	9.66021D−13	2.42808D−11
2^{-7}	1.00782D−04	3.73031D−09	8.67985D−11	2.18475D−13	4.87471D−12
2^{-8}	9.90067D−05	1.89540D−09	1.95806D−11	4.12100D−14	9.67551D−13

Table 6. mean square error (32) for *Example 6*

step size	Maruyama	FRKI	ERKI	SM92	JK95
2^{-4}	4.32409D−02	6.59613D−04	4.71199D−04	6.66800D−05	1.34709D−04
2^{-5}	3.58256D−02	1.51500D−04	5.92752D−05	8.48084D−06	1.43696D−05
2^{-6}	3.35928D−02	4.48836D−05	1.11286D−05	1.61323D−06	1.62352D−06
2^{-7}	3.07276D−02	1.52125D−05	2.21840D−06	2.85247D−07	2.09213D−07
2^{-8}	3.04019D−02	6.40606D−06	5.09955D−07	6.75905D−08	3.08444D−08

From the results in the simulations, we can conclude that in many cases the 4-stage order 1.5 RK scheme developed here is worth the extra computation burden that it involve.

5. Conclusions

For Itô stochastic differential equation (1), an efficient Runge-Kutta (RK) type approximation scheme has been developed. In the proposed RK scheme, both drift and dispersion functions are evaluated by 4 kinds of terms to achieve order 1.5 accuracy without using the derivatives of the coefficients functions. The results on the error evaluation are as follows: *Theorem 1* gives the evaluations of local and global errors of the RK scheme. This theorem guarantees that the RK scheme is of order 1.5 accuracy. The asymptotic efficiency of the numerical solution is also established in *Theorem 2*. The validity of this numerical approximation scheme was verified through the simulation results.

Appendix. (derivation of the order conditions in *Lemma 1*)

The conditions in *Lemma 1* are derived by comparing the coefficients of terms in the Itô-Taylor expansion (3) and the Taylor expansion of the numerical scheme (4). The order 1.5 Taylor expansion of the numerical solution $\bar{x}_n$ around $\bar{x}_n$ at the point $t = t_n$ is given as follows:

$$\begin{aligned}
\bar{x}_{n+1} = \bar{x}_n &+ \Delta W_n[g]_n \sum \bar{b}_i + \Delta\tilde{W}_n[g]_n \sum \tilde{b}_i \\
&+ \sqrt{\nu h}[g]_n \sum \hat{b}_i + h[f]_n \sum b_i + \Delta W_n^2[g_x(g)]_n \sum \bar{b}_i\bar{c}_i \\
&+ \Delta W_n \sqrt{\nu h}[g_x(g)] \sum \bar{b}_i\hat{c}_i + \frac{\Delta\tilde{W}_n}{\sqrt{3}}\sqrt{\nu h}[g_x(g)]_n \sum \tilde{b}_i\hat{c}_i \\
&+ \left(\sqrt{\nu h}\right)^2 [g_x(g)]_n \sum \hat{b}_i\hat{c}_i + h\Delta W_n[f_x(g)]_n \sum b_i\bar{c}_i \\
&+ h\frac{\Delta\tilde{W}_n}{\sqrt{3}}[f_x(g)]_n \sum b_i\tilde{c}_i + h\Delta W_n[g_x(f)]_n \sum \bar{b}_i c_i \\
&+ \Delta W_n^3[g_x(g_x(g))]_n \sum \bar{b}_i\bar{a}_{ij}\bar{c}_j + \Delta W_n^2\sqrt{\nu h}[g_x(g_x(g))]_n \sum \bar{b}_i\bar{a}_{ij}\hat{c}_j \\
&+ \Delta W_n(\sqrt{\nu h})^2[g_x(g_x(g))]_n \sum \bar{b}_i\hat{a}_{ij}\hat{c}_j + \frac{\Delta W_n^3}{2}[g_{xx}(g,g)]_n \sum \bar{b}_i\bar{c}_i^2 \\
&+ \frac{\Delta W_n(\sqrt{\nu h})^2}{2}[g_{xx}(g,g)]_n \sum \bar{b}_i\hat{c}_i^2 + \Delta W_n^2\sqrt{\nu h}[g_{xx}(g,g)]_n \sum \bar{b}_i\bar{c}_i\hat{c}_i \\
&+ h\frac{\Delta\tilde{W}_n}{\sqrt{3}}[g_x(f)]_n \sum \tilde{b}_i c_i + \frac{\Delta\tilde{W}_n}{\sqrt{3}}(\sqrt{\nu h})^2[g_x(g_x(g))]_n \sum \tilde{b}_i\hat{a}_{ij}\hat{c}_j \\
&+ \frac{\Delta\tilde{W}_n(\sqrt{\nu h})^2}{2\sqrt{3}}[g_{xx}(g,g))]_n \sum \tilde{b}_i\hat{c}_i^2 + \left(\sqrt{\nu h}\right)^3 [g_x(g_x(g))]_n \sum \hat{b}_i\hat{a}_{ij}\hat{c}_j \\
&+ \frac{(\sqrt{\nu h})^2}{2}[g_{xx}(g,g)]_n \sum \hat{b}_i\hat{c}_i^2 + r(t_n,\,\bar{x}_n),
\end{aligned} \tag{33}$$

where the term $r(t_n,\,\bar{x}_n)$ is the remainder such that $\mathrm{E}\{\,\|h^{-2}r\|^p\,\} \leq M_r < \infty$.

Then the order conditions in *Lemma 1* are derived as follows, where $\Delta W_n[g]_n \to$ (c.1), for example, should be read as follows : Comparing the coefficients of $\Delta W_n[g]_n$ in (3) and (33) yields the condition (c.1).

$\Delta W_n^2[g_x(g)]_n \to$ (c.2), $(\sqrt{\nu h})^2[g_x(g)]_n \to$ (c.3), $h[f]_n \to$ (c.4),
$\Delta W_n^3[g_{xx}(g,g)]_n/2 \to$ (c.5), $\Delta W_n(\sqrt{\nu h})^2[g_{xx}(g,g)]_n/2 \to$ (c.6),
$\Delta W_n^3[g_x(g_x(g))]_n \to$ (c.7), $\Delta W_n(\sqrt{\nu h})^2[g_x(g_x(g))]_n \to$ (c.8),
$h\Delta W_n[g_x(f)]_n \to$ (c.9), $h\Delta W_n[f_x(g)]_n \to$ (c.10),

$h\Delta\tilde{W}_n[f_x(g)]_n/\sqrt{3} \to$ (c.11), $h\Delta\tilde{W}_n[g_x(f)]_n/\sqrt{3} \to$ (c.12),
$\Delta\tilde{W}_n(\sqrt{\nu h})^2[g_{xx}(g,g)]_n \to$ (c.13), $\Delta W_n^2\sqrt{\nu h}[g_x(g_x(g))]_n \to$ (c.14),
$\Delta W_n\sqrt{\nu h}[g_x(g)]_n \to$ (c.15), $\Delta W_n^2\sqrt{\nu h}[g_{xx}(g,g)]_n \to$ (c.16),
$\Delta\tilde{W}_n[g]_n \to$ (c.17), $\Delta\tilde{W}_n\sqrt{\nu h}[g_x(g)]_n/\sqrt{3} \to$ (c.18),
$\Delta\tilde{W}_n(\sqrt{\nu h})^2[g_x(g_x(g))]_n/\sqrt{3} \to$ (c.19), $\sqrt{\nu h}[g]_n \to$ (c.20),
$(\sqrt{\nu h})^3[g_x(g_x(g))]_n \to$ (c.21), $(\sqrt{\nu h})^2[g_{xx}(g,g)]_n/2 \to$ (c.22).

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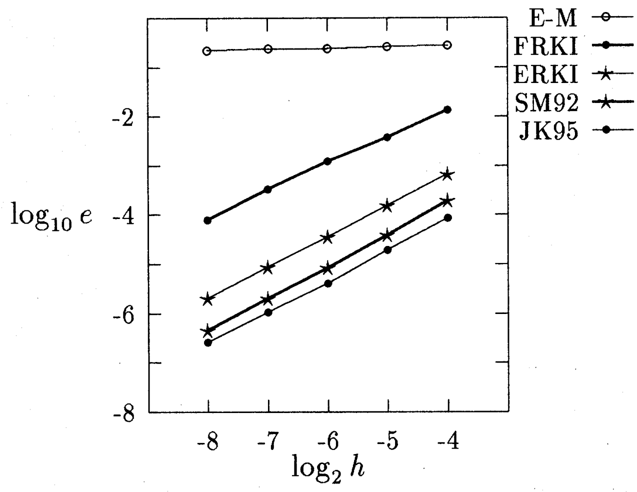

Fig.1 mean square error for Example 1

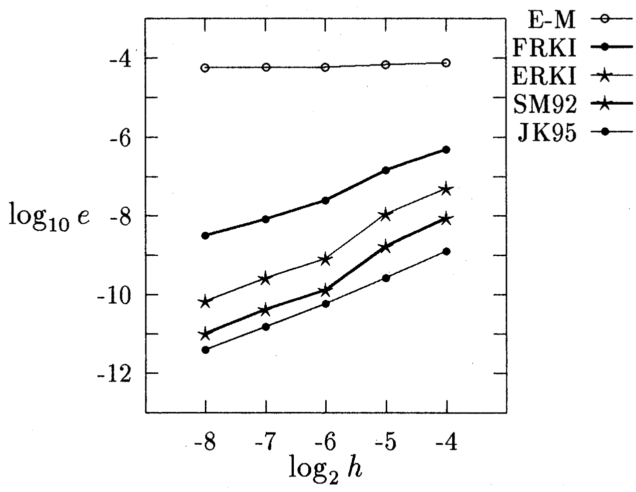

Fig.4 mean square error for Example 4

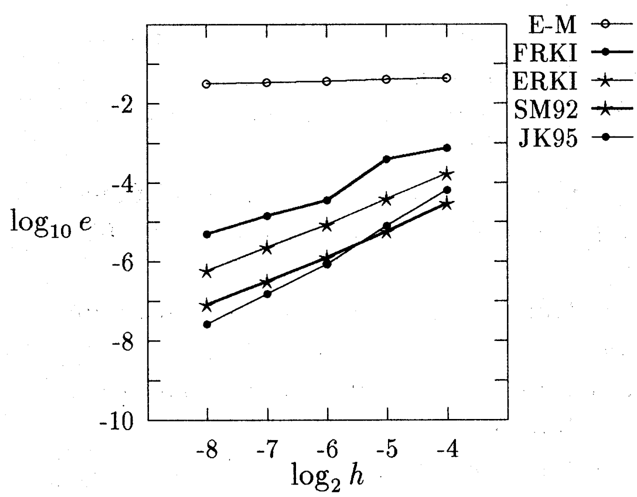

Fig.2 mean square error for Example 2

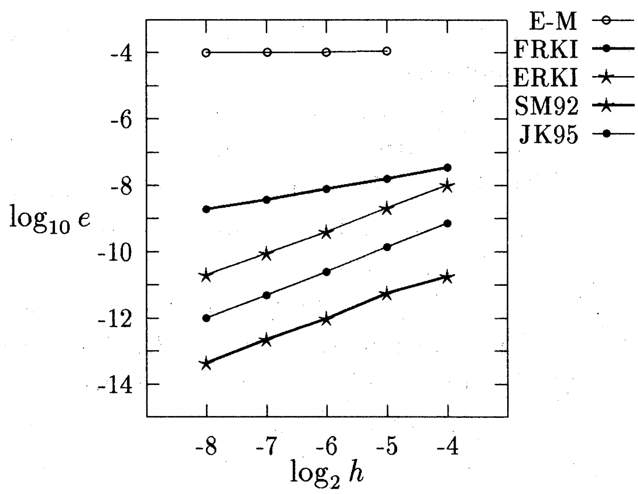

Fig.5 mean square error for Example 5

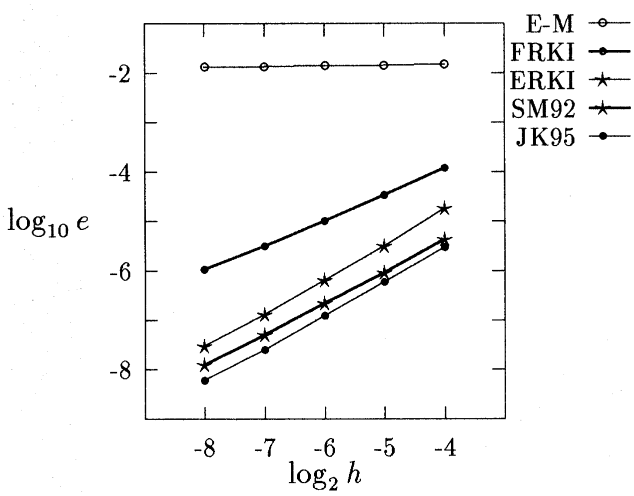

Fig.3 mean square error for Example 3

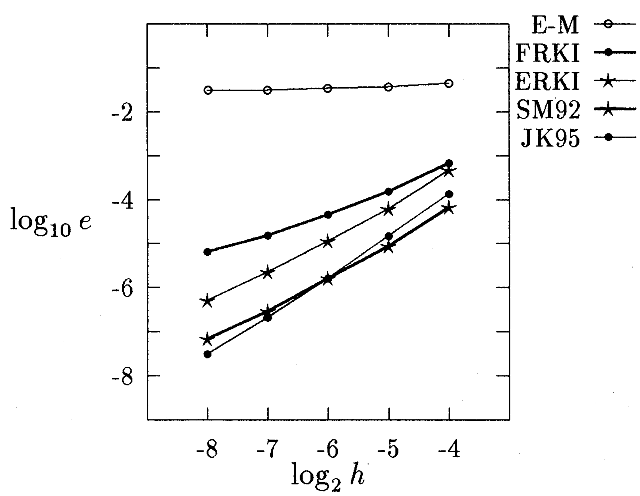

Fig.6 mean square error for Example 6